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ON THE GRAPH REPRESENTATION OF THE $3n + 1$ CONJECTURE

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Abstract. In this study, a graph (tree) representation of the $3n+1$ conjecture has been provided, and some properties of this tree have been demonstrated. These properties allow for a better understanding and interpretation of the results generated by the conjecture's procedure. By utilizing the graph representation in the Quaternary System, the orbits of numbers can be determined more easily.

Keywords: Collatz conjecture, infinite graph, quaternary number system, binary tree.
AMS Subject Classification: 11A99, 11B37, 11K31, 05C63, 05C90.
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1 Introduction

The $3n + 1$ hypothesis was first proposed by Lothar Collatz in 1937 (Lagarias, 1985). This hypothesis is also known by various names such as the Collatz Problem, Ulam's Theorem, Kakutani's Problem, Thwaites' Theorem, Hasse Algorithm, Syracuse Problem and the Hailstone Sequence (Lagarias, 2023).

The hypothesis is based on the following ***COL*** procedure (Lagarias, 1985):

COL1. Choose any positive integer n.
COL2. If n is even, divide it by 2.
COL3. If n is odd, multiply it by 3 and add 1.
COL4. Continue the iteration until the resulting number reaches 1.

The $3n + 1$ hypothesis asserts that "Applying the COL Procedure to all positive integers will eventually lead to the result reaching 1".

Let's calculate the number 5 step by step using the COL Procedure:

5 is odd $\rightarrow$ Apply $3n + 1$ operation $\rightarrow$ Result is 16.
16 is even $\rightarrow$ Apply operation $n/2$ $\rightarrow$ Result is 8.
8 is even $\rightarrow$ Apply operation $n/2$ $\rightarrow$ Result is 4.
4 is even $\rightarrow$ Apply operation $n/2$ $\rightarrow$ Result is 2.
2 is even $\rightarrow$ Apply operation $n/2$ $\rightarrow$ Result is 1.

COL Procedure iteration reached the number 1; terminate the operation. In this example, the number 5 was processed through the COL Procedure. It reached the number 1 in a total of 5 steps.

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The Collatz Conjecture has been the subject of various studies from the past to the present day. Articles have been published in journals, conferences have been held, theses have been written, and research has been conducted at private institutes and universities (Lagarias, 1985), (Lagarias, 2023), (Chamberland, 2010).

The most significant recent work on the Collatz Conjecture belongs to Terence Tao. In his article published in 2019, Tao proved using probability that nearly all Collatz sequences are bounded by any function that diverges to infinity (Tao, 2022).

Today, the computation of the Collatz Conjecture continues to be carried out by computers. Projects such as BOINC and Science United are being conducted at the University of California. These projects are supported by the National Science Foundation (Sonntag, 2020).

As of 2020, the conjecture has been checked by computers for all initial values up to approximately $2^{68} \approx 2.95 \times 10^{20}$ (Barina, 2021).

This study first introduces the Collatz Procedure, presents the $3n + 1$ hypothesis, provides a summary of previous research on this topic, and shows the latest results from computer experiments. The subsequent sections delve into the mathematical representation of the problem, showcase examples, present the known graph representation so far, and propose a new graph representation specifically for odd numbers. In the subsequent sections, the proposed graphical representation is elaborated based on four modules, with a focus on the mathematical formulas underlying them. The encoding of the proposed graphs in the Quaternary Number System is provided. Additionally, some properties of the proposed graphs are demonstrated utilizing the Quaternary Number System.

2 The Mathematical Representation of the $3n + 1$ Problem

Let's first discuss the concept of function composition:

- Consider sets A, B, C and transformations $f : A \to B$, $g : B \to C$. The addition of g to f transformation is denoted as $g \circ f : A \to C$.
- In other words, for any $x \in A$, let $g \circ f(x) = g(f(x))$.
- Let D be a set, h be a transformation $h : D \to D$, and let n be a natural number. The addition of h to itself n times is denoted as $h^n : D \to D$.
- Thus, $h^1 = h$, $h^2 = h \circ h$, $h^{(n+1)} = h \circ h^n$ and more explicitly:

$$h^n = h \circ h \circ h \circ \ldots \circ h \text{ } (n \text{ times})$$

Now, let's consider the f function performed by the Collatz Procedure: Let $N = \{1, 2, \ldots\}$. For all $\forall n \in N$, the f function is defined as follows (Lagarias, 1985).

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n + 1 & \text{if } n \equiv 1 \pmod 2 \end{cases} \tag{1}$$

$$f^k = \underbrace{f \circ f \circ f \circ \ldots \circ f}_{k \text{ times}} \tag{2}$$

Collatz Conjecture: For every $n \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $f^k(n) = 1$, which means: For all $\forall n \in \mathbb{N}$ there exists a $\exists k \in \mathbb{N}$ such that $f^k(n) = 1$.

Example: $6 \to 3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$.
In the example above, for $n = 6$, it takes $k = 8$ steps to reach 1.

Now, let's consider the sequences calculated for numbers from 1 to 10:
COL(1) = { 1 },
COL(2) = { 2, 1 },
COL(3) = { 3, 10, 5, 16, 8, 4, 2,1 },
COL(4) = { 4, 2, 1 },
COL(5) = { 5, 16, 8, 4, 2, 1 },
COL(6) = { 6, 3, 10, 5, 16, 8, 4, 2, 1 },
COL(7) = { 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 },
COL(8) = { 8, 4, 2, 1 },
COL(9) = { 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 },
COL(10) = { 10, 5, 16, 8, 4, 2, 1 }

Sometimes, reaching 1 can take a long time. For example, starting with 27, it takes 111 steps to reach 1:

27 → 82 → 41 → 124 → 62 → 31 → 94 → 47 → 142 → 71 → 214 → 107 → 322 → 161 → 484 → 242 → 121 → 364 → 182 → 91 → 274 → 137 → 412 → 206 → 103 → 310 → 155 → 466 → 233 → 700 → 350 → 175 → 526 → 263 → 790 → 395 → 1186 → 593 → 1780 → 890 → 445 → 1336 → 668 → 334 → 167 → 502 → 251 → 754 → 377 → 1132 → 566 → 283 → 850 → 425 → 1276 → 638 → 319 → 958 → 479 → 1438 → 719 → 2158 → 1079 → 3238 → 1619 → 4858 → 2429 → 7288 → 3644 → 1822 → 911 → 2734 → 1367 → 4102 → 2051 → 6154 → 3077 → 9232 → 4616 → 2308 → 1154 → 577 → 1732 → 866 → 433 → 1300 → 650 → 325 → 976 → 488 → 244 → 122 → 61 → 184 → 92 → 46 → 23 → 70 → 35 → 106 → 53 → 160 → 80 → 40 → 20 → 10 → 5 → 16 → 8 → 4 → 2 → 1 (111 steps).

3 Graph Representation of the COL Procedure

The subsequent sections of the article will be explained using graph terminology (Anderson, 2024).

The operations in the COL Procedure can be visually represented in the following graph shape (Tuncer & Kurum, 2022):

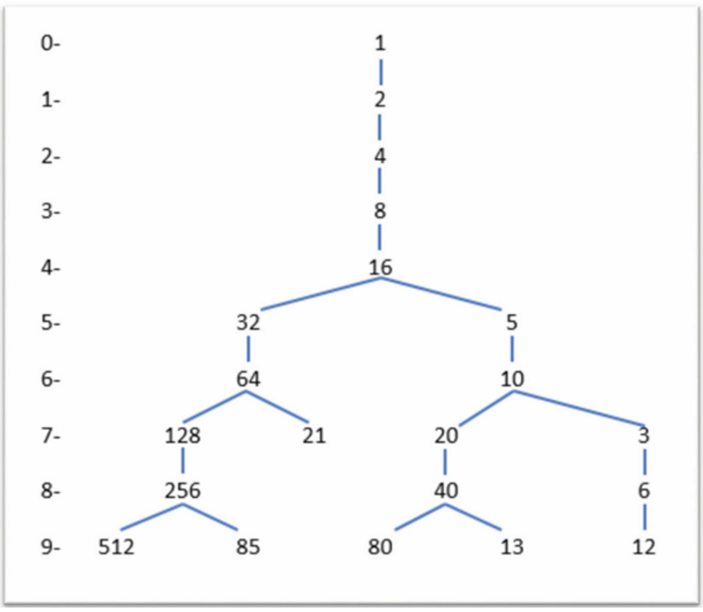

Figure 1: Collatz tree

This is a Rooted Binary Tree with the root being "1". The map graph showing the orbits of numbers in the Collatz Procedure is shown in Figure 2:

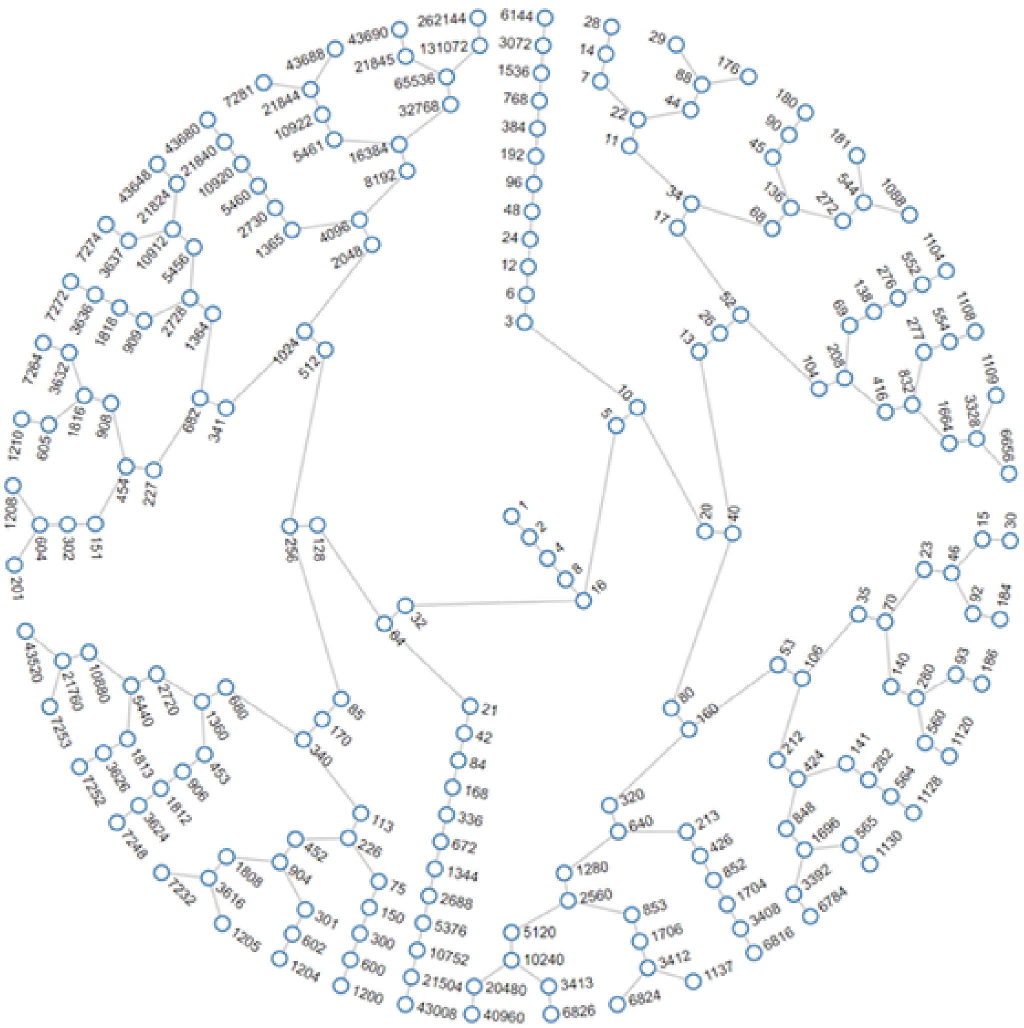

Figure 2: Map graph showing the orbits of numbers in the Collatz Procedure (Davies, 2012)

This graph is also a Rooted Binary Tree (COL Tree) with the root being "1".

In the above graph, if we remove even numbers, we obtain the following Rooted Infinite Collatz tree in odd numbers (COLO) Tree.

In this and subsequent graphs, the roots of trees and sub-trees are colored in GREEN, leaves are in RED, and other vertices are in BLUE.

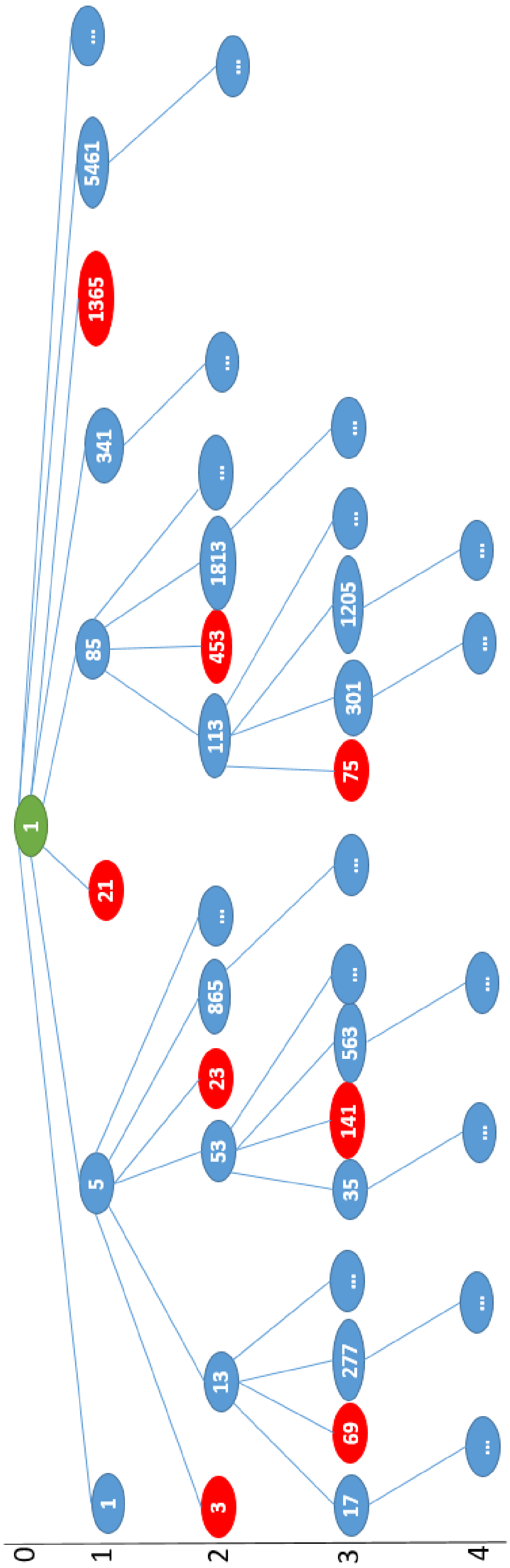

Figure 3: COLO Tree showing the orbits of odd numbers in the Collatz Procedure.

In this tree, every parent at each level has an infinite number of children in the next level. As seen from the diagram, there is a connection between siblings in this tree:

Proposition 1: In the COLO tree, for each height k $(k = 1, 2, \ldots)$, there is a connection between the right sibling $(a^k_{(i+1)})$ and the left sibling $(a^k_{(i)})$ as follows $(i = 2, 3, \ldots)$:

$$a^k_{(i+1)} = 4a^k_{(i)} + 1 \tag{3}$$

Proof. Let's take an arbitrary $a^k_{(i)}$ and its adjacent $a^k_{(i+1)}$ elements from the COLO Tree at level k. Assume that $3a^k_{(i)} + 1 = a^{(k-1)}_j$, which means a^k_i is a child of $a^{(k-1)}_j$. If we can show that $a^k_{(i+1)}$ defined by formula (3) is also a child of $a^{(k-1)}_j$, we will prove Proposition 1. To do this, we need to show that $3a^k_{(i+1)} + 1$ can be reduced to $a^{(k-1)}_j$, i.e., $(3a^k_{i+1} + 1 \to a^{(k-1)}_j)$.

$$3a^k_{(i+1)} + 1 = 3(4a^k_i + 1) + 1 = 12a^k_i + 4 = 4(3a^k_i + 1) = 4 \cdot a^{(k-1)}_j \to a^{(k-1)}_j$$

Thus, $a^k_{(i+1)}$ is also a child of $a^{(k-1)}_j$ and the right sibling of a^k_j. We can observe this property in the Subtrees below as well: □

Proposition 2: In the above COLO Tree and Subtrees, numbers divisible by 3 form the leaves, which means they do not continue in odd numbers.

Proof. We can denote the peak points of odd numbers divisible by 3 in the COLO tree as $a_j = 3(2i - 1)$, $(i = 1, 2, \ldots)$. The children of these peak points in even numbers are then given by $3 \cdot (2i - 1) \cdot 2^s = 3 \cdot 2l$. For these peak points to have children in odd numbers, a_m in $3 \cdot a_m + 1 = 3 \cdot 2l$ must be an odd number. This implies that $3(2l - a_m) = 1$, which is not possible. □

Proposition 3: In the above COLO Tree and Subtrees, there exists a number with the desired length of orbit.

Proof. According to Proposition 2, the orbits starting from 1 in the COLO tree end in numbers divisible by 3. Let one of these numbers be a_{ik}. According to Proposition 1, this number has a sibling calculated as $a^k_{(i+1)} = 4a^k_i + 1$. Since a^k_i is divisible by 3, $a^k_{(i+1)}$ is not divisible by 3, which means it has children and we can continue the orbit through $a^k_{(i+1)}$.

Note: If a^k_i is a leaf, then its siblings expressed as $a^k_{(i+3)}, a^k_{(i+6)}, \ldots, a^k_{(i+3t)}$ also become leaves. However, the peak points expressed as $a^k_{(i+3t-2)}$ and $a^k_{(i+3t-1)}$ between these peak points are not leaves. This means they have children in odd numbers, and these orbits can continue.

□

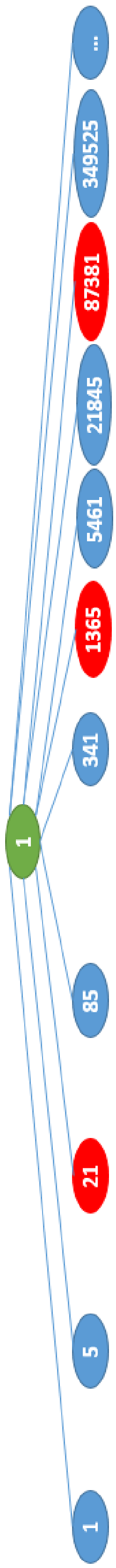

Figure 4: Subtree with root “1”

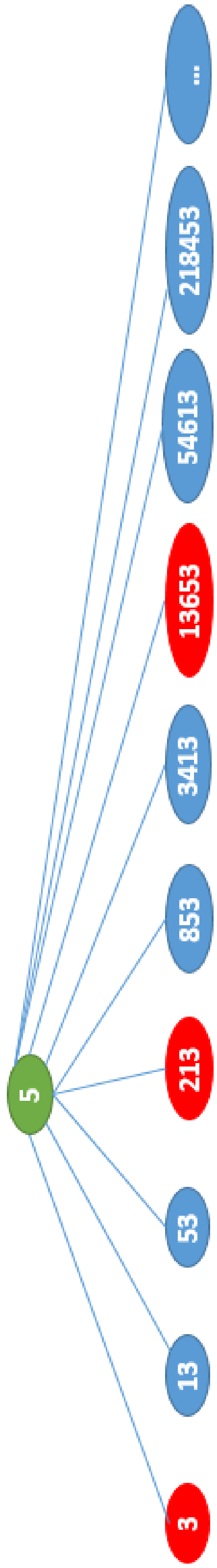

Figure 5: Subtree with root “5”

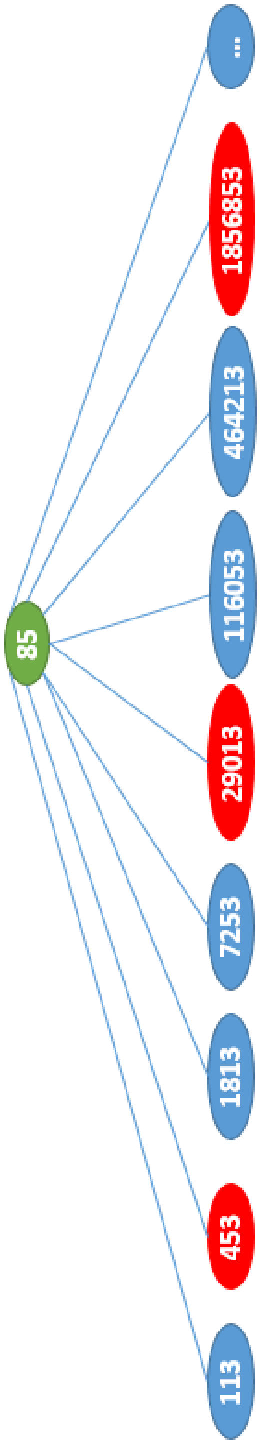

Figure 6: Subtree with root “85”

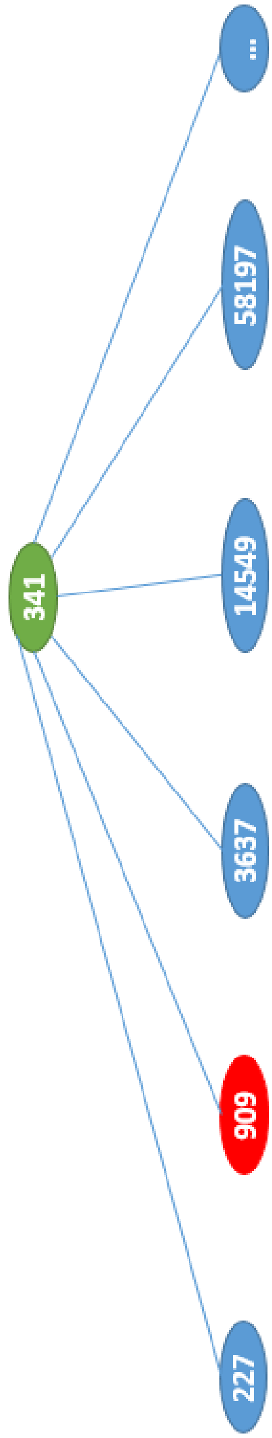

Figure 7: Subtree with root “341”

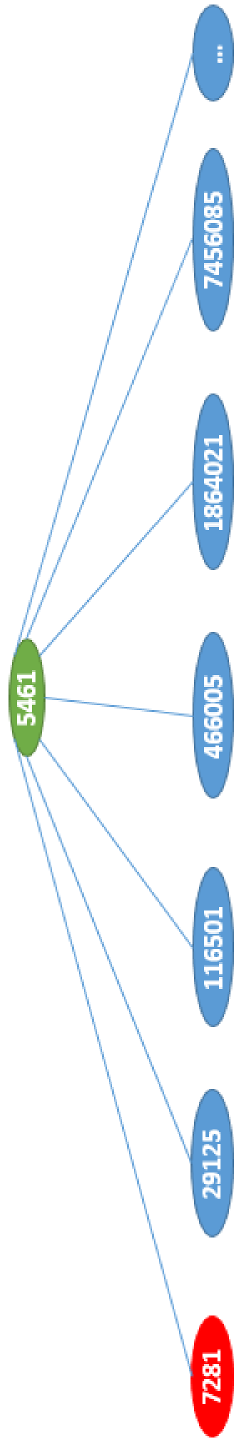

Figure 8: Subtree with root “5461”

4 Mathematical Representation of the $3n+1$ Problem According to its 4 Modules

In (Nuriyeva & Nuriyev, 2017) it has been shown that the following formula is correct for computing the $f(n)$ function:

Theorem 1. *For each $n \in \mathbb{N}$, the $\varphi(n)$ function defined below yields the same results as the $f(n)$ function:*

$$\varphi(n) = \begin{cases} n/4 & \text{if } n \equiv 0 \pmod 4 \\ n-k & \text{if } n \equiv 1 \pmod 4 \\ n/2 & \text{if } n \equiv 2 \pmod 4 \\ n+2\overline{k} & \text{if } n \equiv 3 \pmod 4 \end{cases} \tag{4}$$

Here, $k = \left\lfloor \frac{n}{4} \right\rfloor, \quad \bar{k} = k+1 = \left\lceil \frac{n}{4} \right\rceil$

Proof. Let's demonstrate that we can find the values determined by the $\varphi(n)$ function given in formula (4) by applying the f function for the given n.

$$\begin{aligned} & n \equiv 0 \pmod 4 \\ & \Rightarrow n = 4k \\ & \implies f(n) = \frac{n}{2} = \frac{4k}{2} = 2k \\ & \Rightarrow f(f(n)) = f(2k) = \frac{2k}{2} = k = \frac{n}{4} = \varphi(n) \end{aligned}$$

$$\begin{aligned} & n \equiv 2 \pmod 4 \\ & \Rightarrow n = 4k+2 \\ & \Rightarrow f(n) = \frac{n}{2} = \frac{4k+2}{2} = 2k+1 = n/2 = \varphi(n) \end{aligned}$$

$$\begin{aligned} & n \equiv 1 \pmod 4 \\ & \Rightarrow n = 4k+1 \\ & \Rightarrow f(n) = 3n+1 = 3(4k+1)+1 = 12k+4 = 2(6k+2) \\ & \Rightarrow f(f(n)) = f(n)/2 = 2(6k+2)/2 = 6k+2 = 2(3k+1) \\ & \Rightarrow f(f(f(n))) = f(n)/2 = 2(3k+1)/2 = 3k+1 = (4k+1)-k = n-k = \varphi(n) \end{aligned}$$

$$\begin{aligned} & n \equiv 3 \pmod 4 \\ & \Rightarrow n = 4k+3 \\ & \Rightarrow f(n) = 3n+1 = 3(4k+3)+1 = 12k+10 = 2(6k+5) \\ & \Rightarrow f(f(n)) = f(n)/2 = 2(6k+5)/2 = 6k+5 = (4k+3)+2k+2 \\ & \Rightarrow n+2(k+1) = n+2\overline{k} = \varphi(n) \end{aligned}$$

□

5 Representation of the $3n+1$ Problem in the Quaternary Number System

As seen from equation (4), there is no multiplication operation with '3' here; only division by 2 and 4, multiplication by 2, subtraction, and addition operations are used. The proposed representation allows calculating the f function in the quaternary number system, enabling the explanation of many of its properties.

- $3 \to 5 \to 4 \to 2 \to 1$
- $7 \to 11 \to 17 \to 13 \to 10 \to 5 \to 4 \to 2 \to 1$
- $15 \to 23 \to 35 \to 53 \to 40 \to 20 \to 10 \to 5 \to 4 \to 2 \to 1$

Below, the representations of Figures 3-8 in the quaternary number systems are provided. As shown in Figures 9 - 14 below, there is the following connection between siblings in this tree:

Proposition 5: In the tree Collatz tree in odd numbers in the quaternary system (COLOQ) for each height k ($k = 1, 2, \ldots$), the right sibling ($a^k_{(i+1)}$) is obtained by appending "1" to the quaternary representation of the left sibling ($a^k_{(i)}$) ($i = 2, 3, \ldots$).

Proof. According to Proposition 1, there is the following relationship between the right sibling a^k_{i+1} and the left sibling a^k_i:

$$a^k_{i+1} = 4 \cdot a^k_i + 1$$

In the quaternary system, multiplying by 4 corresponds to multiplying a^k_i's quaternary representation by 10, which means adding a 0 to the end. Adding 1 to this results in adding a "1" to the end of a^k_i's quaternary representation. □

Proposition 6: In the COLOQ tree and subtrees, only some of the leftmost top nodes may have the digit "3" as the last digit in their quaternary representation. The last digit of the quaternary representation of all other nodes is "1" ($k = 1, 2, \ldots; i = 2, 3, \ldots$).

Proof. According to Proposition 5, in the COLOQ tree and its subtrees, the right siblings at each level are obtained by adding 1 to the quaternary representation of the left siblings. Therefore, except for the leftmost sibling at each level, the leftmost digits in the quaternary representation of other siblings consist only of 1's, meaning that the digit 3 can only appear in the leftmost top nodes. □

6 Conclusion

The presented article utilizes the quaternary system representation of the COL Procedure to propose a Rooted Tree structure. This suggested approach allows for a more visual depiction of how the COL Procedure operates. By leveraging this representation, connections between number sequences have been examined, leading to the discovery of certain properties. These properties enable a better understanding and interpretation of the sequences generated by the COL Procedure.

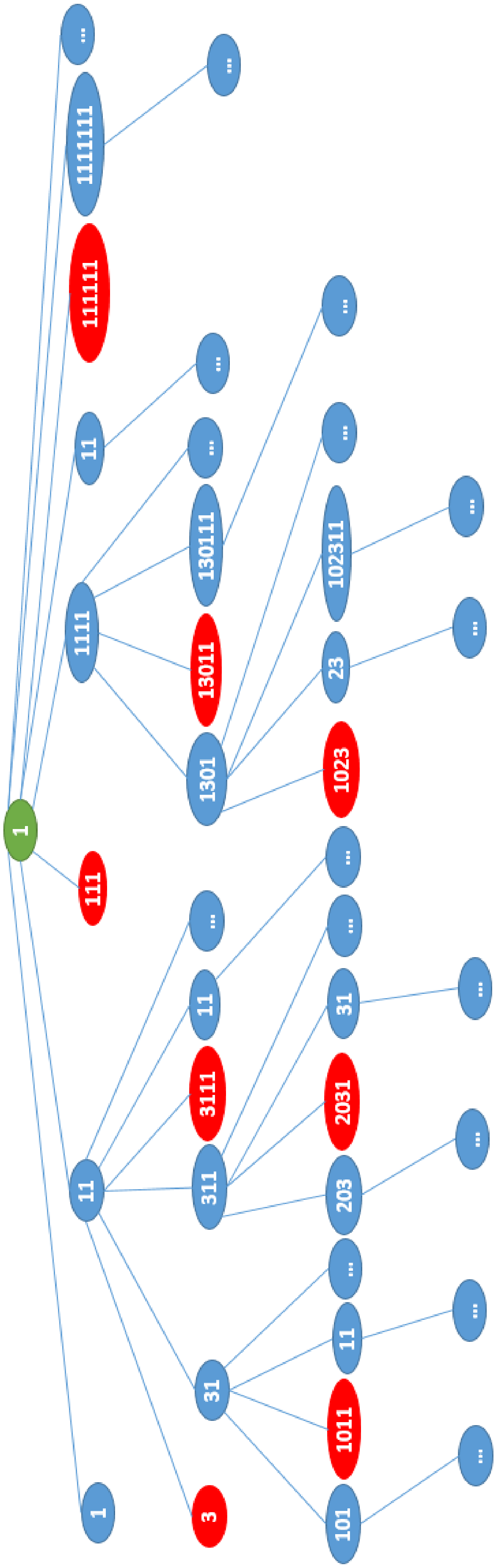

Figure 9: Representation of the COLO Tree in the Quaternary Number System: COLOQ

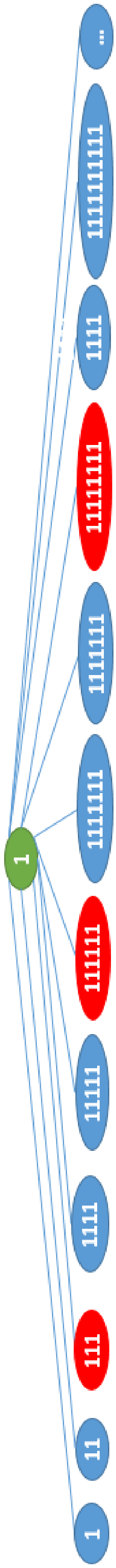

Figure 10: Representation of the Tree with Root “1-(1)” in the Quaternary Number System

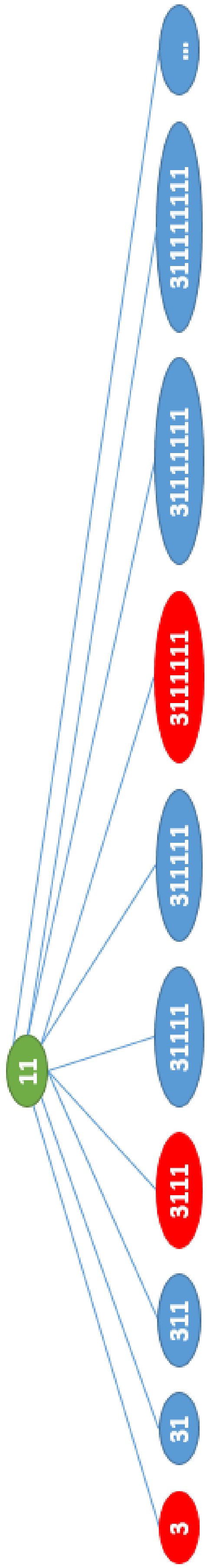

Figure 11: Representation of the Tree with Root “11-(5)” in the Quaternary Number System

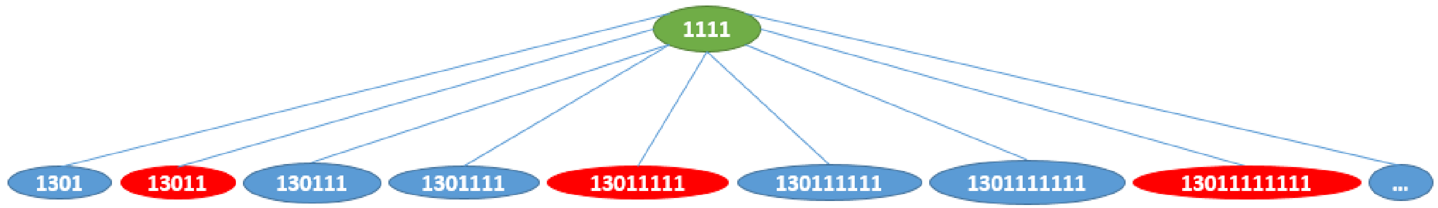

Figure 12: Representation of the Tree with Root "1111-(85)" in the Quaternary Number System

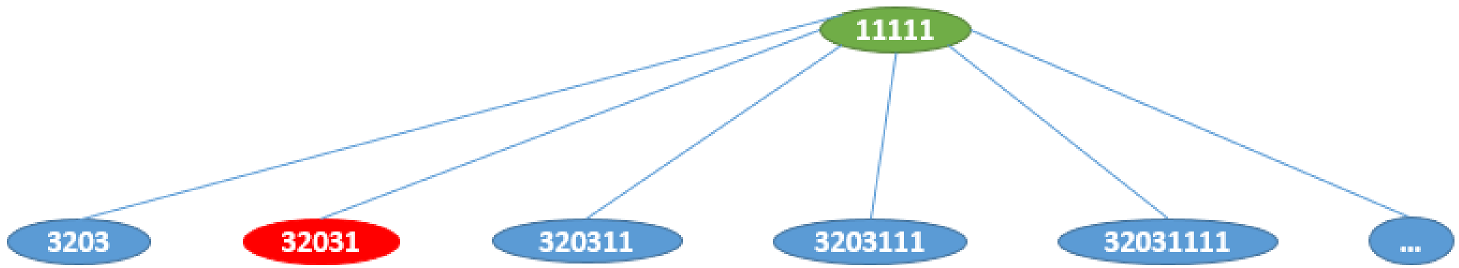

Figure 13: Representation of the Tree with Root "11111-(341)" in the Quaternary Number System

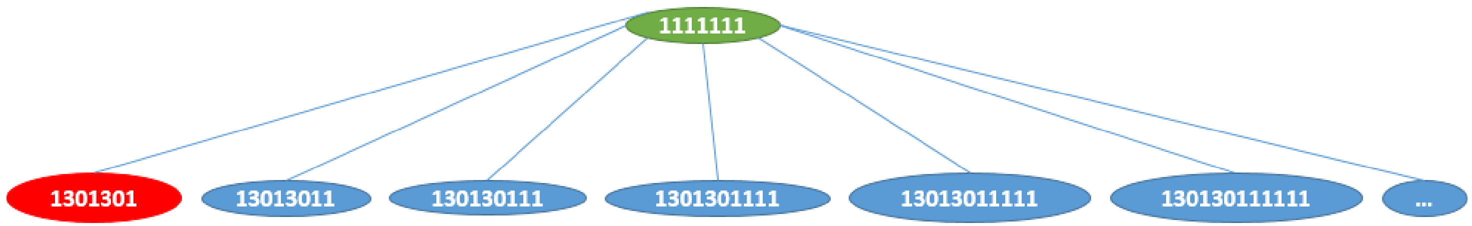

Figure 14: Representation of the Tree with Root "111111-(5461)" in the Quaternary Number System

References

Anderson, J.A. (2024). *Discrete Mathematics with Combinatorics*. Prentice Hall, 928p.

Barina, D. (2021). Convergence verification of the Collatz problem. *The Journal of Supercomputing*, *77*(3), 2681-2688.

Chamberland, M. (2010). A 3x+ 1 survey: number theory and dynamical systems. *The Ultimate Challenge: The 3x, 1*, 57-78.

Davies, J. (2012). Collatz Graph: All Numbers Lead to One. Retrieved from `https://www.jasondavies.com/collatz-graph/`.

Lagarias, J.C. (1985). The 3 x+ 1 problem and its generalizations. *The American Mathematical Monthly*, *92*(1), 3-23.

Lagarias, J.C. (Ed.) (2023). *The Ultimate Challenge: The 3 x+ 1 Problem*. American Mathematical Society.

Nuriyeva, F., Nuriyev, U. (2017). On the 3n+1 Problem. In *30th National Mathematics Symposium (in turkish)*, 2017, pp. 53, Ankara, Türkiye, September 06-09.

Sonntag, J. (2020). What is Collatz Conjecture?. Retrieved from `https://boinc.thesonntags.com/collatz/`.

Tao, T. (2022). Almost all orbits of the Collatz map attain almost bounded values. In *Forum of Mathematics, Pi*, Vol. 10, p. e12. Cambridge University Press.

Tuncer, T., Kurum, H.Y. (2022). A novel Collatz conjecture-based digital image watermarking method. *Cryptologia*, *46*(2), 128-147.